MATH4900E

Team 4 Presentation

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Path in $\mathbb{R}^2$

A path in the plane $\mathbb{R}^2$ is a differentiable function $f : [a, b] \to \mathbb{R}^2$, given by $f(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable functions of t and where $[a, b]$ is some interval in $\mathbb{R}$. The image of an interval $[a, b]$ under a path f is a *curve* in $\mathbb{R}^2$.

Euclidean length

The *Euclidean length* of f is given by the integral

$$length(f) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt,$$

where $\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ is the element of arc-length in $\mathbb{R}^2$.

If we view f as a path into $\mathbb{C}$ instead of $\mathbb{R}^2$ and write $f(t) = x(t) + y(t)i$, we can rewrite the integral as

$$length(f) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b |f'(t)|dt,$$

and represent the standard element of arc-length in $\mathbb{C}$ as

$$|dz| = |f'(t)|dt.$$

Path Integral

Let $\rho : \mathbb{C} \to \mathbb{R}$ be a continuous function. For a differentiable path $f : [a, b] \to \mathbb{C}$, we define the length of f with respect to the element of arc-length $\rho(z)|dz|$ to be the path integral

$$length_\rho(f) = \int_f \rho(z)|dz| = \int_a^b \rho(f(t))|f'(t)|dt.$$

Question: What will happen to the length of a path $f : [a, b] \to \mathbb{C}$ with respect to the element of arc-length $\rho(z)|dz|$ when the domain of f is changed?
i.e. Suppose $h : [\alpha, \beta] \to [a, b]$ is a surjective differentiable function such that $[a, b] = h([\alpha, \beta])$, and construct a new path by taking the composition $g = f \circ h$. How are $length_\rho(f)$ and $length_\rho(g)$ related?

The length of f with respect to $\rho(z)|dz|$ is the path integral

$$\begin{aligned} length_\rho(f) &= \int_f \rho(z)|dz| \\ &= \int_a^b \rho(f(t))|f'(t)|dt, \end{aligned}$$

while the length of g with respect to $\rho(z)|dz|$ is the path integral

$$\begin{aligned} length_\rho(g) &= \int_\alpha^\beta \rho(g(t))|g'(t)|dt \\ &= \int_\alpha^\beta \rho((f \circ h)(t))|(f \circ h)'(t)|dt \\ &= \int_\alpha^\beta \rho(f(h(t)))|(f'(h(t))||h'(t)|dt. \end{aligned}$$

If $h'(t) \geq 0$ for all t in $[\alpha, \beta]$, then

$$
\begin{aligned}
length_\rho(g) &= \int_\alpha^\beta \rho(f(h(t)))|(f'(h(t))||h'(t)|dt \\
&= \int_a^b \rho(f(s))|f'(s)|ds = length_\rho(f).
\end{aligned}
$$

with substitution $s = h(t)$.
If $h'(t) \leq 0$ for all t in $[\alpha, \beta]$, then

$$
\begin{aligned}
length_\rho(g) &= \int_\alpha^\beta \rho(f(h(t)))|(f'(h(t))||h'(t)|dt \\
&= -\int_a^b \rho(f(s))|f'(s)|ds = length_\rho(f).
\end{aligned}
$$

with substitution $s = h(t)$.

So if either $h'(t) \geq 0$ or $h'(t) \leq 0$ for all t in $[\alpha, \beta]$, we have

$$length_\rho(f) = length_\rho(f \circ h),$$

where $f : [a, b] \to \mathbb{C}$ is a piecewise differentiable path and $h : [\alpha, \beta] \to [a, b]$ is differentiable. In this case, we refer to $f \circ h$ as a *reparametriaztion* of f.

Proposition 1

Let $f : [a, b] \to \mathbb{C}$ be a piecewise differentiable path, let $[\alpha, \beta]$ be another interval, and let $h : [\alpha, \beta] \to [a, b]$ be a surjective differentiable function. Let $\rho(z)|dz|$ be an element of arc-length on $\mathbb{C}$. Then

$$length_\rho(f \circ h) \geq length_\rho(f).$$

Let $\rho(z)|dz|$ be an element of arc-length on $\mathbb{H}$ that is a conformal distortion of the standard element of arc-length, so that the length of a piecewise differentiable path $f : [a, b] \to \mathbb{H}$ is given by the integral

$$length_\rho(f) = \int_f \rho(z)|dz| = \int_a^b \rho(f(t))|f'(t)|dt.$$

By the phrase *length is invariant* under the action of $M\ddot{o}b(\mathbb{H})$, for every piecewise differentiable path $f : [a, b] \to \mathbb{H}$ and every element γ of $M\ddot{o}b(\mathbb{H})$, we have

$$length_\rho(f) = length_\rho(\gamma \circ f).$$

Proposition 2

Let γ be a *Möbius* transformation of $\mathbb{H}$. Let $z, z' \in \mathbb{H}$ and let δ be a path from z to z'. Then $length_{\mathbb{H}}(\gamma \circ \delta) = length_{\mathbb{H}}(\delta)$.

Proof.
Let $\gamma(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. It is an easy calculation to check that for any $z \in \mathbb{H}$,

$$|\gamma'(z)| = \frac{ad - bc}{|cz + d|^2}$$

and

$$Im(\gamma(z)) = \frac{ad - bc}{|cz + d|^2} Im(z).$$

Let $\delta : [0,1] \to \mathbb{H}$ be a parametrization of δ. Then by chain rule,

$$\begin{aligned}
length_{\mathbb{H}}(\gamma \circ \delta) &= \int_0^1 \frac{|(\gamma \circ \delta)'(t)|}{Im(\gamma \circ \delta)(t)} dt \\
&= \int_0^1 \frac{|\gamma'(\delta(t))||\delta'(t)|}{Im(\gamma \circ \delta)(t)} dt \\
&= \int_0^1 \frac{ad - bc}{|c\delta(t) + d|^2} |\delta'(t)| \frac{|c\delta(t) + d|^2}{ad - bc} \frac{1}{Im(\delta(t))} dt \\
&= \int_0^1 \frac{|\delta'(t)|}{Im(\delta(t))} dt \\
&= length_{\mathbb{H}}(\delta).
\end{aligned}$$

Since

$$\begin{aligned} length_\rho(\gamma \circ f) &= \int_a^b \rho((\gamma \circ f)(t))|(\gamma \circ f)'(t))|dt \\ &= \int_a^b \rho((\gamma \circ f)(t))|\gamma'(f(t))||f'(t)|dt \end{aligned}$$

and

$$length_\rho(f) = \int_a^b \rho(f(t))|f'(t)|dt,$$

we have

$$\int_a^b \rho(f(t))|f'(t)|dt = \int_a^b \rho((\gamma \circ f)(t))|\gamma'(f(t))||f'(t)|dt$$

for every piecewise differentiable path $f : [a, b] \to \mathbb{H}$ and every element γ of $M\ddot{o}b^+(\mathbb{H})$.

Equivalently, this can be written as

$$\int_a^b (\rho(f(t)) - \rho((\gamma \circ f)(t))|\gamma'(f(t))|)|f'(t)|dt = 0$$

for every piecewise differentiable path $f : [a, b] \to \mathbb{H}$ and every element γ of $M\ddot{o}b^+(\mathbb{H})$.
For an element γ of $M\ddot{o}b^+(\mathbb{H})$, set

$$\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)|,$$

so that the condition on $\rho(z)$ becomes a condition on $\mu_\gamma(z)$, that is

$$\int_f \mu_\gamma(z)|dz| = \int_a^b \mu_\gamma(f(t))|f'(t)|dt = 0$$

for every piecewise differentiable path $f : [a, b] \to \mathbb{H}$ and every element γ of $M\ddot{o}b^+(\mathbb{H})$.

Lemma 3

Let D of an open set of $\mathbb{C}$, let $\mu : D \to \mathbb{R}$ be a continuous function, and suppose that $\int_f \mu(z)|dz| = 0$ for every piecewise differentiable path $f : [a, b] \to D$. Then $\mu \equiv 0$.

Proof

We do by contradiction.
Suppose there exists a point $z \in D$ at which $\mu(z) \neq 0$. Replacing μ by $-\mu$ if necessary, we may assume that $\mu(z) > 0$.
Since μ is continuous, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $U_\delta(z) \subset D$ and $w \in U_\delta(z)$ implies that $\mu(w) \in U_\varepsilon(\mu(z))$, where

$$U_\delta(z) = u \in \mathbb{C} : |u - z| < \delta$$

and

$$U_\varepsilon(t) = s \in \mathbb{R} : |s - t| < \varepsilon.$$

Taking $\varepsilon = \frac{1}{3}|\mu(z)|$, we see that there exists $\delta > 0$ so that $w \in U_\delta(z)$ implies that $\mu(w) \in U_\varepsilon(\mu(z))$. Using the triangle inequality and the fact that $\mu(z) > 0$, this implies that $\mu(w) > 0$ for all $w \in U_\delta(z)$. We now choose a specific non-constant piecewise differentiable path, namely the path $f : [0, 1] \to U_\delta(z)$ given by

$$f(t) = z + \frac{1}{3}\delta t.$$

Observe that $\mu(f(t)) > 0$ for all t in $[0, 1]$, since $f(t) \in U_\delta(z)$ for all t in [0,1]. In particular, we have that $\int_f \mu(z)|dz| > 0$, which gives the desired contradiction.

Hence by the lemma, we have

$$\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)| = 0$$

for every $z \in \mathbb{H}$ and every element γ of $M\ddot{o}b^+(\mathbb{H})$.
We now consider how μ_γ behaves under composition of elements of $M\ddot{o}b^+(\mathbb{H})$.

Let γ and φ be two elements in $M\ddot{o}b^+(\mathbb{H})$.

$$\begin{aligned}\mu_{\gamma\circ\varphi}(z) &= \rho(z) - \rho((\gamma\circ\varphi)(z))|(\gamma\circ\varphi)'(z)| \\ &= \rho(z) - \rho((\gamma\circ\varphi)(z))|\gamma'(\varphi(z))||\varphi'(z)| \\ &= \rho(z) - \rho(\varphi(z))|\varphi'(z)| + \rho(\varphi(z))|\varphi'(z)| \\ &\quad - |\rho((\gamma\circ\varphi)(z))|\gamma'(\varphi(z))||\varphi'(z)| \\ &= \mu_\varphi(z) + \mu_\gamma(\varphi(z))|\varphi'(z)|.\end{aligned}$$

In particular, if $\mu_\gamma \equiv 0$ for every γ in a generating set for $M\ddot{o}b^+(\mathbb{H})$, then $\mu_\gamma \equiv 0$ for every element γ of $M\ddot{o}b^+(\mathbb{H})$.

$M\ddot{o}b(\mathbb{H})$ is generated by elements of the form $m(z) = az + b$ for $a > 0$ and $b \in \mathbb{R}$, $K(z) = \frac{-1}{z}$, and $B(z) = -\overline{z}$.

Note that the elements listed as generators are all elements of $M\ddot{o}b(\mathbb{H})$. Also note that every element of $M\ddot{o}b(\mathbb{H})$ has either the form

$$m(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, or the form

$$n(z) = \frac{a\overline{z} + b}{c\overline{z} + d},$$

where a, b, c, d is purely imaginary and $ad - bc = 1$.

If $c = 0$, then $m(z) = \frac{a}{d}z + \frac{b}{d}$. Since $ad - bc = ad = 1$, we have $\frac{a}{d} = a^2 > 0$.
If $c \neq 0$, then $m(z) = f(K(g(z)))$, where $g(z) = c^2 z + cd$ and $f(z) = z + \frac{a}{c}$.
Note that $B \circ n = m$, where m is an element of $M\ddot{o}b(\mathbb{H})$, we can write $n = B^{-1} \circ m = B \circ m$.

Then we consider a generator $\gamma(z) = z + b$ for $b \in \mathbb{R}$ first. Since $\gamma'(z) = 1$ for every $z \in \mathbb{H}$, the condition imposed on $\rho(z)$ is that

$$0 \equiv \mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)| = \rho(z) - \rho(z+b)$$

for every $z \in \mathbb{H}$ and every $b \in \mathbb{R}$. That is

$$\rho(z) = \rho(z+b)$$

for every $z \in \mathbb{H}$ and every $b \in \mathbb{R}$. In particular, $\rho(z)$ depends only on the imaginary part $y = Im(z)$ of $z = x + iy$.

To see this explicitly, suppose that $z_1 = x_1 + iy$ and $z_2 = x_2 + iy$ have the same imaginary part, and write $z_2 = z_1 + (x_2 - x_1)$. Since $x_2 - x_1$ is real, we have $\rho(z_2) = \rho(z_1)$.
Hence we may view ρ as a real-valued function of the single real variable $y = Im(z)$. Explicitly, consider the real-valued function $r : (0, \infty) \to (0, \infty)$ given by $r(y) = \rho(iy)$, and note that $\rho(z) = r(Im(z))$ for every $z \in \mathbb{H}$.

Next we consider the generator $\gamma(z) = az$ for $a > 0$. Since $\gamma'(z) = a$ for every $z \in \mathbb{H}$, the condition imposed on $\rho(z)$ is that

$$0 \equiv \mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)| = \rho(z) - a\rho(az)$$

for every $z \in \mathbb{H}$ and every $a > 0$. That is,

$$\rho(z) = a\rho(az)$$

for every $z \in \mathbb{H}$ and every $a > 0$. In particular, we have

$$r(y) = ar(ay)$$

for every $y > 0$ and every $a > 0$. Interchanging the roles of a and y, we see that $r(a) = yr(ay)$. Dividing through by y, we obtain

$$r(ay) = \frac{1}{y} r(a).$$

Taking $a = 1$, this yields that

$$r(y) = \frac{1}{y} r(1),$$

and r is completely determined by its value at 1.
Recalling the definition of r, we have the invariance of length under $M\ddot{o}b^{+}(\mathbb{H})$ implies that $\rho(z)$ has the form

$$\rho(z) = r(Im(z)) = \frac{c}{Im(z)},$$

where c is an arbitrary positive constant.

We now take the transformations $K(z) = -\frac{1}{z}$ and $B(z) = -\bar{z}$ into our consideration.
Since $K'(z) = \frac{1}{z^2}$, the condition imposed on $\rho(z)$ is that

$$0 = \mu_K(z) = \rho(z) - \rho(K(z))|K'(z)| = \rho(z) - \rho(-\frac{1}{z})\frac{1}{|z|^2}.$$

Substituting $\rho(z) = \frac{c}{Im(z)}$ and using

$$\rho(-\frac{1}{z}) = \rho(\frac{-\bar{z}}{|z|^2}) = \frac{c|z|^2}{Im(-\bar{z})} = \frac{c|z|^2}{Im(z)},$$

we obtain

$$\rho(z) - \rho(-\frac{1}{z})\frac{1}{|z|^2} = \frac{c}{Im(z)} - \frac{c|z|^2}{Im(z)}\frac{1}{|z|^2} = \frac{c}{Im(z)} - \frac{c}{Im(z)} = 0.$$

Note that $B'(z)$ is not defined. So we cannot check by doing similar calculations like in $K(z)$. Instead we want to show

$$length(B \circ f) = length(f).$$

Note that $B \circ f(t) = -x(t) + iy(t)$. Then $|(B \circ f)'(t)| = |f'(t)|$ and $Im(B \circ f)(t) = y(t) = Im(f(t))$, and so

$$\begin{aligned} length(B \circ f) &= \int_a^b \frac{c}{Im((B \circ f)(t))} |(B \circ f)'(t)| dt \\ &= \int_a^b \frac{c}{Im(f(t))} |f'(t)| dt = length(f). \end{aligned}$$

Therefore we have the following theorem:

Theorem 4

For every positive constant c, the element of arc-length

$$\frac{c}{Im(z)}|dz|$$

on $\mathbb{H}$ is invariant under the action of $M\ddot{o}b(\mathbb{H})$. That is, for every piecewise differentiable path $f : [a, b] \to \mathbb{H}$ and every element γ of $M\ddot{o}b(\mathbb{H})$, we have that

$$length_\rho(f) = length_\rho(\gamma \circ f).$$

However, nothing we have done to this point has given us a way of determining a specific value of c. In fact, it is not possible to specify the value of c using solely the action of $M\ddot{o}b(\mathbb{H})$. To avoid carrying c through all our calculations, we set $c = 1$.

Example

For a real number $\lambda > 0$, let A_λ be the Euclidean line segment joining $-1 + i\lambda$ to $1 + i\lambda$, and let B_λ be the hyperbolic line segment joining $-1 + i\lambda$ to $1 + i\lambda$. Cauculate the lengths of A_λ and B_λ with respect to the element of arc-length $\frac{c}{Im(z)}|dz|$.

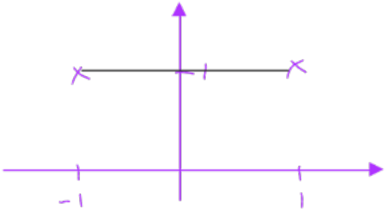
Solution.
We parametrize A_λ by the path $f : [-1, 1] \to \mathbb{H}$ given by $f(t) = t + i\lambda$.Since $Im(f(t)) = \lambda$ and $|f'(t)| = 1$, we see that

$$length(f) = \int_{-1}^{1} \frac{c}{\lambda} dt = \frac{2c}{\lambda}.$$

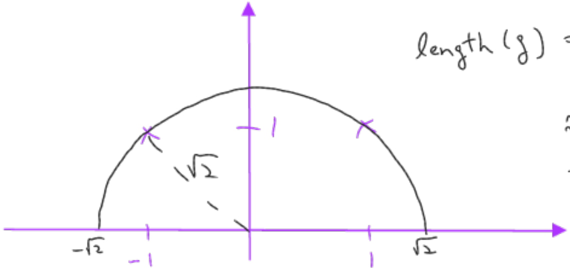
B_λ lies on the Euclidean circle with Euclidean centre 0 and Euclidean radius $\sqrt{1+\lambda^2}$. The Euclidean line segment between 0 and $1+i\lambda$ makes angle θ with the positive real axis, where $cos(\theta) = \frac{1}{\sqrt{1+\lambda^2}}$. So we can parametrize B_λ by the path $g : [\theta, \pi - \theta] \to \mathbb{H}$ given by $g(t) = \sqrt{1+\lambda^2} e^{i\theta}$. Since $Im(g(t)) = \sqrt{1+\lambda^2} sin(\theta)$ and $|g'(t)| = \sqrt{1+\lambda^2}$, we see that

$$length(g) = \int_{\theta}^{\pi-\theta} c\, csc(t) dt = c\, ln[\frac{\sqrt{1+\lambda^2}+1}{\sqrt{1+\lambda^2}-1}].$$

$\lambda = 1$: $(c = 1)$



$$\text{length}(f) = 2c = 2$$



$$\text{length}(g) = c \ln\left(\frac{\sqrt{1+1^2}+1}{\sqrt{1+1^2}-1}\right)$$

$$\approx 1.76\, c$$

$$= 1.76 \; (< 2)$$

Definition 5

For a piecewise differentiable path $f : [a, b] \to \mathbb{H}$, we define the hyperbolic length of f to be

$$length_{\mathbb{H}}(f) = \int_f \frac{1}{Im(z)}|dz| = \int_a^b \frac{1}{Im(f(t))}|f'(t)|dt.$$

Example

Take $0 < a < b$ and consider the path $f : [a, b] \to \mathbb{H}$ given by $f(t) = it$. The image $f([a, b])$ of $[a, b]$ under f is the segment of the positive imaginary axis between ai and bi. Since $Im(f(t)) = t$ and $|f'(t)| = 1$, we see that

$$length_{\mathbb{H}}(f) = \int_f \frac{1}{Im(z)}|dz| = \int_a^b \frac{1}{t}dt = ln[\frac{b}{a}].$$

Proposition 6

Let $f : [a, b] \to \mathbb{H}$ be a piecewise differentiable path. Then the hyperbolic length $length_{\mathbb{H}}(f)$ of f is finite.
Note: this provides a way to estimate an upper bound for the hyperbolic length of a path in $\mathbb{H}$.

Proof

There exists a constant $B > 0$ so that the image $f([a, b])$ of $[a, b]$ under f is contained in the subset

$$K_B = \{z \in \mathbb{H} | Im(z) \geq B\}$$

of $\mathbb{H}$. Given that $f([a, b])$ is contained in K_B, we can estimate the integral giving the hyperbolic length of f. We first note that by the definition of piecewise differentiable, there is a partition P of $[a, b]$ inito subintervals

$$P = [a = a_0, a_1], [a_1, a_2], ..., [a_n, a_{n+1} = b]$$

so that f is differentiable on each subinterval $[a_k, a_{k+1}]$.

In particular, its derivativ f' is continuous on each subinterval. By the extreme value theorem for a continuous function on a closed interval, there then exists for each k a number A_k so that

$$|f'(t)| \leq A_k \, \forall t \in [a_k, a_{k+1}].$$

Let A be the maximum of $A_0, \ldots, A_n$. Then we have

$$length_H(f) = \int_a^b \frac{1}{Im(f(t))} |f'(t)| dt \leq \int_a^b \frac{1}{B} A \, dt = \frac{A}{B}(b - a),$$

which is finite.

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Definition 7

A *metric* on a set X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying three conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

Definition 8

Let X be a metric space with metric d. We say that (X, d) is a path metric space if for each pair of points x and y of X we have

$$d(x, y) = inf\{length(f) : f \in \Gamma[x, y]\},$$

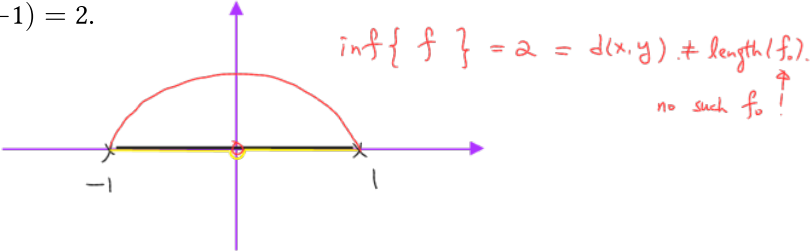
and for each pair of points x and y of X, there exists a distance realizing path in $\Gamma[x, y]$, which is a path f in $\Gamma[x, y]$ satisfying

$$d(x, y) = length(f).$$

Example

$(\mathbb{C}, n)$ is a path metric space while $(\mathbb{C} - \{0\}, n)$ is not, where $n(x, y) = |x - y|$ on $\mathbb{C}$ and $\mathbb{C} - \{0\} = X$ respectively.

Consider two points 1 and -1 in (X, n). The Euclidean line segment joining 1 to -1 passes through 0, and so is not a path in X. Every other path joining 1 to -1 has length strictly greater than $n(1, -1) = 2$.



Theorem 9

$(\mathbb{H}, d_{\mathbb{H}})$ is a path metric space. Moreover, the distance realizing path in $\Gamma[x, y]$ is a parametrization of the hyperbolic line segment joining x to y.
(Proof: Omitted.)

Proposition 10

For every element γ of $M\ddot{o}b(\mathbb{H})$ and for every pair x and y of points of $\mathbb{H}$, we have

$$d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(\gamma(x), \gamma(y)).$$

Note: We call γ is an isometry of $\mathbb{H}$.

Proof.
Observe that $\gamma \circ f : f \in \Gamma[x, y] \subset \Gamma[\gamma(x), \gamma(y)]$. To see this, take a path $f : [a, b] \to \mathbb{H}$ in $\Gamma[x, y]$, so that $f(a) = x$ and $f(b) = y$. Since $\gamma \circ f(a) = \gamma(x)$ and $\gamma \circ f(b) = \gamma(y)$, we have $\gamma \circ f$ lies in $\Gamma[\gamma(x), \gamma(y)]$.

Since $length_{\mathbb{H}}(f)$ is invariant under the action of $Möb(\mathbb{H})$, we have

$$length_{\mathbb{H}}(\gamma \circ f) = length_{\mathbb{H}}(f)$$

for every path f in $\Gamma[x, y]$, and

$$\begin{aligned} d_{\mathbb{H}}(\gamma(x), \gamma(y)) &= inf\{length_{\mathbb{H}}(g) : g \in \Gamma[\gamma(x), \gamma(y)]\} \\ &\leq inf\{length_{\mathbb{H}}(\gamma \circ f) : f \in \Gamma[x, y]\} \\ &\leq inf\{length_{\mathbb{H}}(f) : f \in \Gamma[x, y]\} = d_{\mathbb{H}}(x, y). \end{aligned}$$

Since γ in invertible and γ^{-1} is an element of $M\ddot{o}b(\mathbb{H})$, we may repeat the argument to see that

$$\{\gamma^{-1} \circ g | g \in \Gamma[\gamma(x), \gamma(y)]\} \subset \Gamma[x, y],$$

and hence

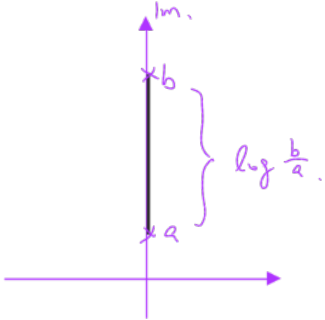
$$\begin{aligned} d_{\mathbb{H}}(x, y) &= inf\{length_{\mathbb{H}}(f) : f \in \Gamma[x, y]\} \\ &\leq inf\{length_{\mathbb{H}}(\gamma^{-1} \circ g) : g \in \Gamma[\gamma(x), \gamma(y)]\} \\ &\leq inf\{length_{\mathbb{H}}(g) : g \in \Gamma[\gamma(x), \gamma(y)]\} = d_{\mathbb{H}}(\gamma(x), \gamma(y)). \end{aligned}$$

Therefore we have $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(\gamma(x), \gamma(y))$ and this completes the proof.

We now proceed to calculate the geodesics in $\mathbb{H}$. Geodesics is the paths of shortest distance in $\mathbb{H}$. In this section we will show that the imaginary axis is a geodesic. Then we will claim that any vertical straight line and any circle meeting the real axis orthogonally is also a geodesic. In here we denote $\mathcal{H}$ the set of semi-circles orthogonal to $\mathbb{R}$ and the vertical lines in the upper half-plane $\mathbb{H}$.

Proposition 11

Let $a < b$. Then the hyperbolic distance between ia and ib is $log\frac{b}{a}$. Moreover, the vertical line joining ia to ib is the unique path between ia and ib ith length $log\frac{b}{a}$; any other path from ia to ib has length strictly greater than $log\frac{b}{a}$.



Proof

Let $\delta(t) = it$, $a \leq t \leq b$. Then δ is a path from ia to ib. Clearly $|\delta'(t)| = 1$ and $Im(\delta(t) = t$ so that

$$length_{\mathbb{H}}(\delta) = \int_a^b \frac{1}{t} dt = log \frac{b}{a}.$$

Now let $\delta(t) = x(t) + iy(t) : [0, 1] \to \mathbb{H}$ be any path from ia to ib. Then

$$\begin{aligned} length_{\mathbb{H}}(\delta) &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \\ &\geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \\ &\geq \int_0^1 \frac{y'(t)}{y(t)} dt \\ &= log y(t)|_0^1 \\ &= log \frac{b}{a}. \end{aligned}$$

Note:
For the first inequality, equality holds when $x'(t) = 0$. This happens when $x(t)$ is a constant, that is we have a path δ which is a vertical line joining ia to ib.
For the second inequality, equality holds when $|y'(t)| = y'(t)$. This happens when $y'(t)$ is positive for all t. This means the path δ travels 'straight up' the imaginary axis from ia to ib without doubling back on itself.
Therefore, we have shown that $length_{\mathbb{H}}(\delta) \geq log\frac{b}{a}$ in general. Equality holds when δ is the vertical path joining ia to ib.

Proposition 12

Let $H \in \mathcal{H}$. $\gamma(H) \in \mathcal{H}$.

Proof.
Recall a vertical line or a circle with a real centre in $\mathbb{C}$ is given by an equation of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Let

$$w = \gamma(z) = \frac{az + b}{cz + d}.$$

Then

$$z = \frac{dw - b}{-cw + a}.$$

Then we have

$$\alpha\left(\frac{dw-b}{-cw+a}\right)\left(\frac{d\overline{w}-b}{-c\overline{w}+a}\right)+\beta\left(\frac{dw-b}{-cw+a}\right)+\beta\left(\frac{d\overline{w}-b}{-c\overline{w}+a}\right)+\gamma=0.$$

Hence

$$\begin{aligned}\alpha(dw-b)(d\overline{w}-b)+\beta(dw-b)(-c\overline{w}+a)\\ +\beta(d\overline{w}-b)(-cw+a)+\gamma(-cw+a)(-c\overline{w}+a)=0.\end{aligned}$$

Expanding this gives

$$\begin{aligned}(\alpha d^2-2\beta cd+\gamma c^2)w\overline{w}+(-\alpha bd+\beta ad+\beta bc-\gamma ac)w\\ +(-\alpha bd+\beta ad+\beta bc-\gamma ac)\overline{w}+(\alpha b^2-2\beta ab+\gamma a^2)=0.\end{aligned}$$

This has the form $\alpha' w\overline{w}+\beta' w+\beta'\overline{w}+\gamma'$ with $\alpha',\beta',\gamma'\in\mathbb{R}$, which is the equation of either a vertical line or a circle with real centre.

Lemma 13

Let $H \in \mathcal{H}$. Then there exists $\gamma \in M\ddot{o}b(\mathbb{H})$ such that γ maps H bijectively to the imaginary axis.

Proof

Case 1: If H is the vertical line $Re(z) = a$ then the translation $z \mapsto z - a$ is a *Möbius transformation* of $\mathbb{H}$ that maps H to the imaginary axis $Re(z) = 0$.
Case 2: Let H be a semi-circle with end points $\zeta_-, \zeta_+ \in \mathbb{R}$, $\zeta_- < \zeta+$.
First note that, the imaginary axis is characterised as the unique element of H with end-points at 0 and ∞. Consider the map

$$\gamma(z) = \frac{z - \zeta_+}{z - \zeta_-}.$$

As $-\zeta_- + \zeta_+ > 0$, this is a *Möbius transformation* of $\mathbb{H}$. Note that $\gamma(H) \in H$. Clearly $\gamma(\zeta_+) = 0$ and $\gamma(\zeta_-) = \infty$, so $\gamma(H)$ must be the imaginary axis.

Lemma 14

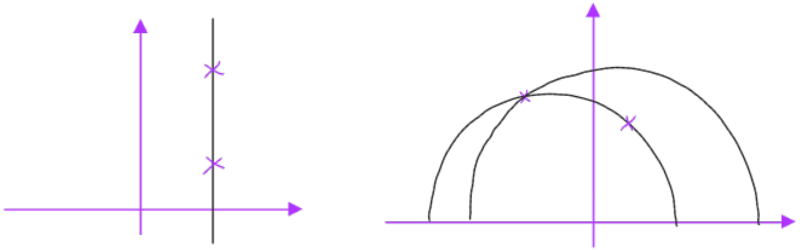
Let $H \in \mathcal{H}$ and let $z_0 \in H$. Then there exists a *Möbius transformation* of $\mathbb{H}$ that maps H to the imaginary axis and z_0 to i.

Proof

Proceed as in the proof of the previous Lemma, we obtain a *Möbius transformation* $\gamma_1 \in Möb(\mathbb{H})$ mapping H to the imaginary axis. Now $\gamma_1(z_0)$ lies on the imaginary axis. For any $k > 0$, the *Möbius transformation* $\gamma_2(z) = kz$ maps the imaginary axis to itself. For a suitable choice of $k > 0$ it maps $\gamma_1(z_0)$ to i. The composition $\gamma = \gamma_2 \circ \gamma_1$ is the required *Möbius transformation* of $\mathbb{H}$.

Theorem 15

The geodesics in $\mathbb{H}$ are the semi-circles orthogonal to the real axis and the vertical straight lines. Moreover, given any two points in $\mathbb{H}$ there exists a unique geodesic passing through them.



Proof

Let $z, z' \in \mathbb{H}$. Then we can always find a unique element of $H \in \mathcal{H}$ containing z, z'. If z and z' have the same real part then H will be a vertical straight line, otherwise H will be a semi-circle with a real centre. Let δ be any path from z to z'.

Apply *Möbius transformation* $\gamma \in M\ddot{o}b(\mathbb{H})$ using Lemma 13, $\gamma(z), \gamma(z')$ lie on the imaginary axis. Then $\gamma \circ \delta$ is a path from $\gamma(z)$ to $\gamma(z')$. We have $length_{\mathbb{H}}(\delta) = length_{\mathbb{H}}(\gamma \circ \delta)$ by Proposition 2.

The imaginary axis is the unique geodesic passing through $\gamma(z)$ and $\gamma(z')$ by Proposition 11. Hence $length_{\mathbb{H}}(\gamma \circ \delta)$ achieves its infimum when $\gamma \circ \delta$ is the arc of imaginary axis form $\gamma(z)$ to $\gamma(z')$.
Hence $length_{\mathbb{H}}(\delta)$ achieves infimum when $\gamma \circ \delta$ is the imaginary axis from $\gamma(z)$ to $\gamma(z')$. This is when δ is the image under γ^{-1} of the imaginary axis from $\gamma(z)$ to $\gamma(z')$. As $\gamma^{-1} \in M\ddot{o}b(\mathbb{H})$, it follows from Proposition 12 that δ is an arc of straight line or semi-circle with real centre passing through z, z'.

We now have a method to calculate the hyperbolic distance between a pair of points in $\mathbb{H}$ in theory. That is, given a pair of points x and y in $\mathbb{H}$, find or construct an element γ of $M\ddot{o}b(\mathbb{H})$ so that $\gamma(x) = i\mu$ ad $\gamma(y) = i\lambda$ both lie on the positive imaginary axis. Then determine the values of μ and λ to find the hyperbolic distance

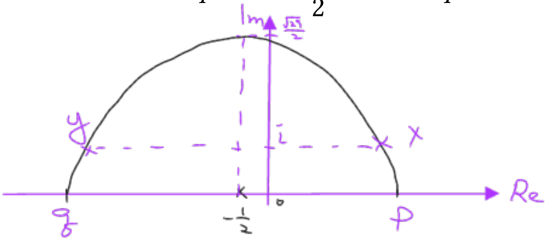
$$d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(\mu i, \lambda i) = |ln[\frac{\lambda}{\mu}]|.$$

Note that here we use the absolute value, as we have made no assumption about whether $\lambda < \mu$ or $\mu < \lambda$.

Example

Consider the two points $x = 2 + i$ and $y = -3 + i$. The hyperbolic line l passing through x and y lies in the Euclidean circle with Euclidean centre $-\frac{1}{2}$ and Euclidean radius $\frac{\sqrt{29}}{2}$. In particular, the endpoints at infinity of l are

$$p = \frac{-1+\sqrt{29}}{2} \text{ and } q = \frac{-1-\sqrt{29}}{2}.$$

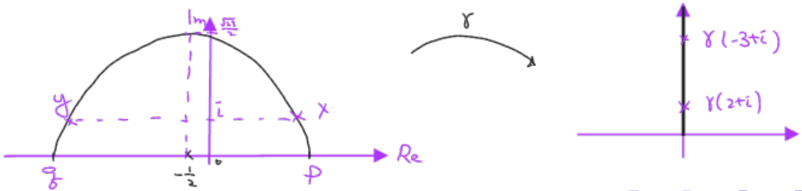


Set $\gamma(z) = \frac{z-p}{z-q}$. The determinant γ is $p - q > 0$, so γ lies in $M\ddot{o}b^+(\mathbb{H})$. Since by construction γ takes the endpoints at infinity of l to the endpoints at infinity of the positive imaginary axis, namely 0 and ∞, we see that γ takes l to the positive imaginary axis. We see that

$$\gamma(2+i) = \frac{2+i-p}{2+i-q} = \frac{p-q}{(2-q)^2+1} i$$

and

$$\gamma(-3+i) = \frac{-3+i-p}{-3+i-q} = \frac{p-q}{(3+q)^2+1} i.$$



So we have

$$\begin{aligned} d_{\mathbb{H}}(2+i,-3+i) &= d_{\mathbb{H}}(\gamma(2+i),\gamma(-3+i)) \\ &= ln[\frac{(2-q)^2+1}{(3+q)^2+1}] \\ &= ln[\frac{58+10\sqrt{29}}{58-10\sqrt{29}}] \end{aligned}$$

which is approximately 3.294.

We transfer the hyperbolic element of arc-length from $\mathbb{H}$ to $\mathbb{D}$ by making the following observation. For any piecewise differnetiable path $f : [a, b] \to \mathbb{D}$, the composition $n \circ f : [a, b] \to \mathbb{H}$ is a piecewise differentiable path into $\mathbb{H}$. We know how to calculate the hyperbolic length of $n \circ f$, namely by integrating the hyperbolic element of arc-length $\frac{1}{Im(z)}|dz|$ on $\mathbb{H}$ along $n \circ f$. So, we define the hyperbolic length of f in $\mathbb{D}$ by

$$length_{\mathbb{D}}(f) = length_{\mathbb{H}}(n \circ f).$$

$$n(z) = \frac{\frac{i}{\sqrt{2}} z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} z - \frac{i}{\sqrt{2}}}$$

$$= -\frac{1}{\sqrt{2}} \frac{iz + 1}{z + i}$$

Theorem 16

The hyperbolic length of a piecewise differentiable path $f : [a, b] \to \mathbb{D}$ is given by

$$length_{\mathbb{D}}(f) = \int_f \frac{2}{1 - |z|^2} |dz|.$$

Proof

We consider the map $h : \mathbb{H} \to \mathbb{D}$ defined by

$$h(z) = \frac{z - i}{iz - 1}.$$

Note that h maps $\mathbb{H}$ bijectively to $\mathbb{D}$, as well as $\partial\mathbb{H}$ to $\partial\mathbb{D}$ bijectively. Let $g(z) = h^{-1}(z)$. Then g maps $\mathbb{D}$ to $\mathbb{H}$ and has the formula

$$g(z) = \frac{-z + i}{-iz + 1}.$$

Let $\delta : [a, b] \to \mathbb{D}$ be a (parametrisation of a) path in $\mathbb{D}$. Then $g \circ \delta : [a, b] \to \mathbb{H}$ is a path in $\mathbb{H}$. The length of $g \circ \delta$ is given by:

$$length_{\mathbb{H}}(g \circ \delta) = \int_a^b \frac{|(g \circ \delta)'(t)|}{Im(g \circ \delta(t))} dt = \int_a^b \frac{|g'(\delta(t))||\delta'(t)|}{Im(g \circ \delta(t))} dt$$

by chain rule. We have

$$g'(z) = \frac{-2}{(-iz + 1)^2}$$

and

$$Im(g(z)) = \frac{1 - |z|^2}{|-iz + 1|^2}.$$

Hence

$$length_{\mathbb{H}}(g \circ \delta) = \int_a^b \frac{2}{1 - |\delta(t)|^2} |\delta'(t)| dt.$$

Then

$$length_{\mathbb{D}}(\delta) = \int_a^b \frac{2}{1 - |\delta(t)|^2} |\delta'(t)| dt = \int_\delta \frac{2}{1 - |z|^2}.$$

The distance between two points $z, z' \in \mathbb{D}$ is defined by taking the length of the shortest path between them. We denote $d_{\mathbb{D}}(z, z') = inf\{length_{\mathbb{D}}(\delta) | \delta$ is a piecewise continuously differentiable path from z to $z'\}$.
As we have used h to transfer the distance function on $\mathbb{H}$ to a distance function on $\mathbb{D}$, we have

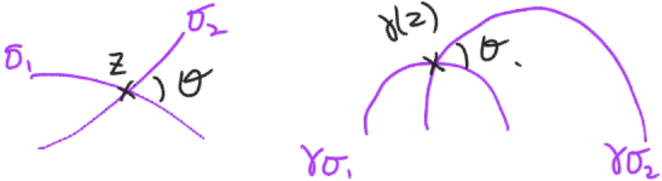
$$d_{\mathbb{D}}(h(z), h(w)) = d_{\mathbb{H}}(z, w).$$

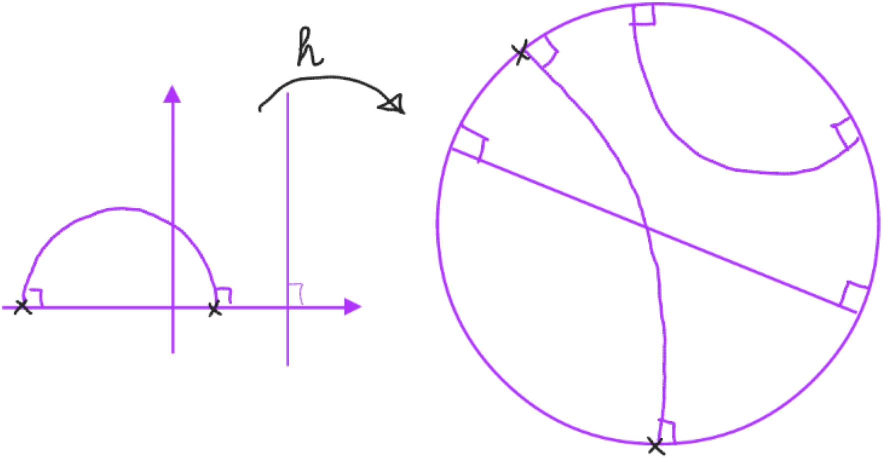
Proposition 17

The geodesics in the Poincare disc are the diameters of $\mathbb{D}$ and the arcs of the circles in $\mathbb{D}$ that meet $\partial\mathbb{D}$ at right-angles.

Proof

One can show that h is conformal, i.e. h preserves angles. Using the characterisation of lines in $\mathbb{C}$ to circles and lines in $\mathbb{C}$. Recall that h maps $\partial\mathbb{H}$ to $\partial\mathbb{D}$. Recall that the geodesics in $\mathbb{H}$ are the arcs of the circles and lines that meet $\partial\mathbb{H}$ orthogonally. As h is conformal, the image in $\mathbb{D}$ of a geodesic in $\mathbb{H}$ is a circle or line that meets $\partial\mathbb{D}$ orthogonally.

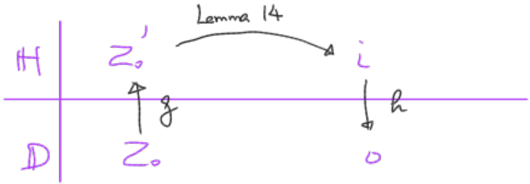



h

In the upper half-plane model $\mathbb{H}$ we often map a geodesic H to the imaginary axis and a point z_0 on that geodesic to the point i. The following is the analogue of the result in the Poincare disc model.

Proposition 18

Let H be a geodesic in $\mathbb{D}$ and let $z_0 \in H$. Then there exists a *Möbius transformation* of $\mathbb{D}$ that maps H to the real axis and z_0 to 0.



References

Hyperbolic geometry, by James W. Anderson, Springer, 1999.
(Chapter 3.1-3.5)
Lecture notes by C. Walkden
(Chapter 3-6)

CLASSIFICATION OF MOBIUS MAP

PREREQUISITE KNOWLEDGE

Topology

MATH3070

One-point compactification, Homeomorphism, Connectedness

Algebra

MATH2070, MATH3030

Group, Quotient Space, Matrix, Isomorphism

Complex calculus

MATH2230, MATH4060

Derivative of Analytic Function

Hyperbolic Function

ONE-POINT COMPACTIFICATION

- Let X be a topological space with topology J such that X is locally compact and Hausdorff.
- Then there exists topological space $X^* = X \cup \{\infty\}$ such that X^* is compact and open sets in X are also open sets in X^*
- X^* is called the one-point compactification of X

CONNECTEDNESS

- Let (X, J) be topological space and $W \subset X$
- W *is connected* if there are no disjoint, non-empty open set U, V such that $W = U \cup V$

Remark: The connectedness of any set is preserved by homeomorphism (or continuous map)

QUOTIENT SPACE

- Let S be a non-empty set and $\sim$ be an equivalence relation
- $[x] = \{y \in S: x \sim y\}$ is called the equivalence class of x
- Then the set of all equivalence class in S is the quotient set (space)

MATRIX GROUP

- Let n be positive integer and F be a field
- General Linear Group $GL(n, F) = \{A \in F^{n\times n}: |A| \neq 0\}$
- Special Linear Group $SL(n, F) = \{\mathrm{A} \in F^{n\times n}: |A| = 1\}$

GROUP ISOMORPHISM

- Let $(G,*)$ and $(H,+)$ be two groups
- $f: G \to H$ is a group isomorphism if
  - $f(a * b) = f(a) + f(b)$
  - f is bijective

CLASSIFICATION: "WHICH SUBSETS OF THE OBJECT SHARE SOME COMMON CHARACTERISTICS?"

Linear Algebra

Classification Vector Space according to dimension

Rank-nullity Theorem

Group Theory

Classification of finite simple group

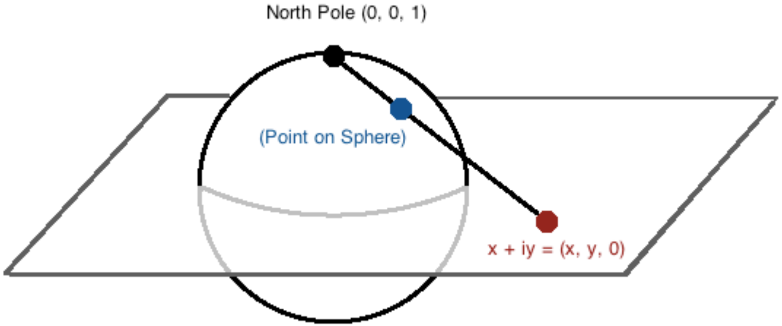
Geometry

Classification of Isometries in Euclidean Plane

Classification of Isometries in Hyperbolic Plane

TARGET SPACE: $\mathbb{C}_\infty$

- $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$
- One-point Compactification of $\mathbb{C}$
- Homeomorphic to Riemann Sphere



MOBIUS GROUP $Mob(\mathbb{C}_\infty)$

- Mobius Transformation is a map $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$ in the form of $f(z) = \frac{az+b}{cz+d}$ with $\mathrm{a, b, c, d} \in \mathbb{C}\ ad - bc \neq 0$
- Basic Properties Mentioned Before
  - The set of all Mobius map forms a group $Mob(\mathbb{C}_\infty)$ with operation defined as combination of map
  - Angle Preserving
  - Act transitively on ordered triples of distinct complex number
  - Map circle to circle
  - Homeomorphism

MATRIX REPRESENTATION OF $Mob(\mathbb{C}_\infty)$

- Intuition:
  - The map is determined by 4 coefficient a, b, c, d
  - Represent them by 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the combination of map can become operation on matrix!
  - The operation is actually matrix multiplication ! [Check it as an exercise]
- Question: $Mob(\mathbb{C}_\infty) \cong GL(2, \mathbb{C})$?

LIMITATION OF $GL(2,\mathbb{C})$

- Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ $in\ GL(2,\mathbb{C})$
- Obviously, they are different elements in $GL(2,\mathbb{C})$
- But they represent same Mobius map $\frac{az+b}{cz+d} = \frac{kaz+kb}{kcz+kd}$
- We solve this problem by using $SL(2,\mathbb{C})$ instead
- The ambiguity in matrix representation is reduced to only differ by $\pm$ signs.

Remark: Although the representation in $SL(2,\mathbb{C})$ is not unique, it is concrete enough to tackle with many problems.

$Mob(\mathbb{C}_\infty) \cong PSL(2, \mathbb{C})$

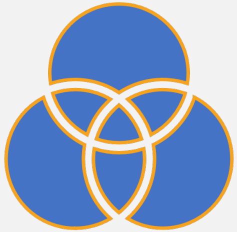
- To solve the ambiguity in $\pm$ signs, we introduce the relation $\sim$: $A \sim -A$
- The quotient set of $SL(2, \mathbb{C})$ under $\sim$ is denoted as $PSL(2, \mathbb{C})$
- It is not hard to observe that both groups are isomorphic to each other

BEFORE NEXT SECTION…

- Every map $\frac{az+b}{cz+d} \in Mob(\mathbb{C}_\infty)$ can be represented by $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ $with\ a'd' - b'c' = 1$
- The composition of map is just multiplication of matrix
- Some properties of matrix are vitally important for classifying $Mob(\mathbb{C}_\infty)$

CONJUGATE AND INVARIANT

- Definition:
  - $Let\ A, B \in Mob(\mathbb{C}_\infty)$ A is conjugate of B if $\exists S \in Mob(\mathbb{C}_\infty)$ such that $A = SBS^{-1}$
- Example:
- $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & -i \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2i & i \\ i & 0 \end{pmatrix}$
- Equivalence Relation !
  - Exercise: Verify symmetricity, transitivity and reflexive property



FIXED POINT UNDER CONJUGATION

- Theorem 1:
- Suppose $A = SBS^{-1}$ and z_0 is a fixed point of B. Then $S(z_0)$ is a fixed point of A.
- Proof:
- $A\big(S(z_0)\big) = SBS^{-1}(z_0) = SB(z_0) = S(z_0)$
- By Theorem 1, number of fixed point is invariant under Conjugation.

TRACE IS INVARIANT

- Theorem 2:
- Suppose A, B are conjugate. Then $Tr(A) = Tr(B)$
- Proof:
- Suppose $A = SBS^{-1}$
- $Tr(A) = Tr(SBS^{-1}) = Tr(SS^{-1}B) = Tr(B)$

TRACE AND NUMBER OF FIXED POINTS

- Theorem 3
- $Let\ A \in Mob(\mathbb{C}_\infty)$ with $A \neq id$
- A has one or two fixed points. A has one fixed point if and only if $Tr(A) = \pm 2$
- Proof:
- Consider Quadratic Equation $\frac{az+b}{cz+d} = z$ ↔️ $cz^2 + (d-a)z - b = 0$
- So the number of roots of the equation is determined by $(d-a)^2 + 4bc = (d+a)^2 - 4 = Tr(T)^2 - 4$
- Hence, A has one fixed point when $Tr(T) = \pm 2$ and having two fixed points otherwise

SUMMARY

- Some maps in $Mob(\mathbb{C}_\infty)$ are equivalent in terms of conjugate relation
- In that equivalence class, they sharing some common characteristics:
  - Same Trace
  - Same number of fixed points
- Mapping of Fixed Point under Conjugation
- The relation between Trace and Numbers of Fixed Points

CASE 1: ONE FIXED POINT z_0

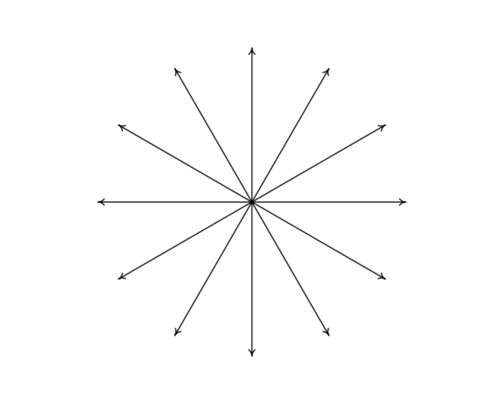
- Through Conjugation $S = \frac{1}{z-z_0}$, we can map the fixed point to ∞
- Suppose $T \in Mob(\mathbb{C}_\infty)$ with ∞ as only fixed point
- Represent T as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C})$
- $T(\infty) = \infty$ implies $c = 0$
- $ad - bc = 1$ implies $d = \frac{1}{a}$
- $Tr(T) = \pm 2$ implies $a = \pm 1$
- Hence, $T(z) = z \pm b$ [Behave like translation in $\mathbb{R}^n$]
- We call this type of transformation $Parabolic$

CASE 2: TWO FIXED POINT z^+, z^-

- Through conjugation $S = \frac{z-z^+}{z-z^-}$, we can map the fixed points to $0\ and\ \infty$
- Suppose $T \in Mob(\mathbb{C}_\infty)$ has $0\ and\ \infty$ as fixed points
- $T(\infty) = \infty$ implies $c = 0$
- $ad - bc = 1$ implies $d = \frac{1}{a}$
- $T(0) = 0$ implies $b = 0$
- Hence $T(z) = a^2 z$
- Denoted $\lambda = a^2$, we can classify them according nature of λ

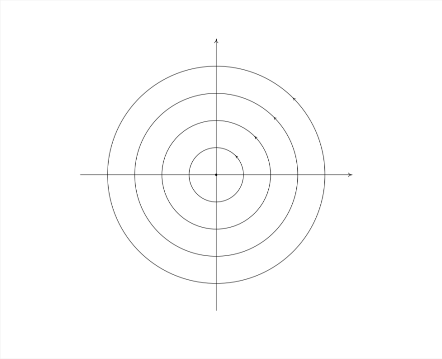
HYPERBOLIC

- If $\lambda \in \mathbb{R}\ and\ |\lambda| \neq 1$ Then T is called $Hyperbolic$
- $T(z) = \lambda z$ behave like scaling in $\mathbb{R}^n$ because $\lambda \in \mathbb{R}$



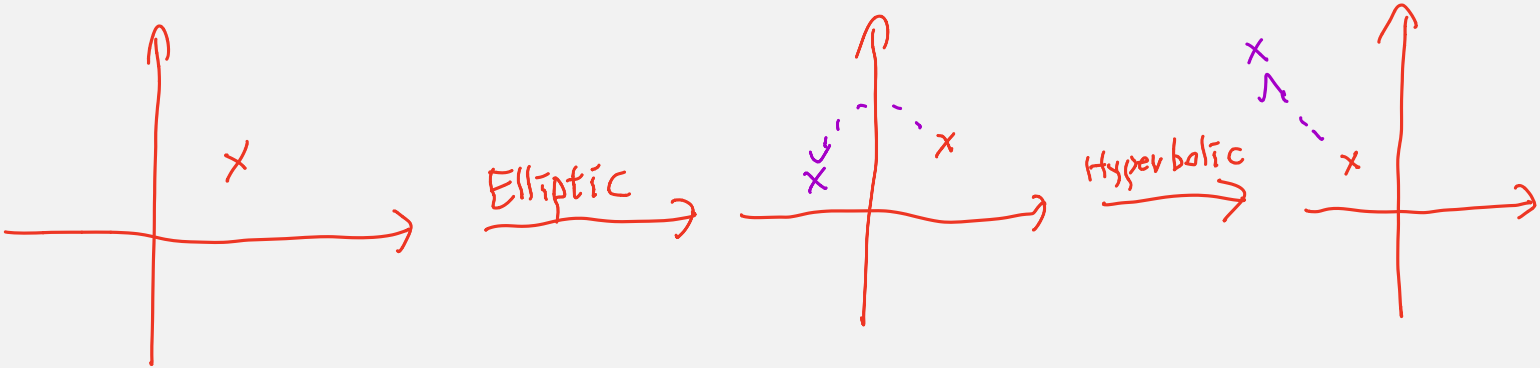
ELLIPTIC

- If $|\lambda| = 1$ Then we call T to be $elliptic$
- Using Poler Form $z = e^{\arg(z)i}$, $T(z) = \lambda z$ is actually rotation of $\arg(\lambda)$ about the origin.

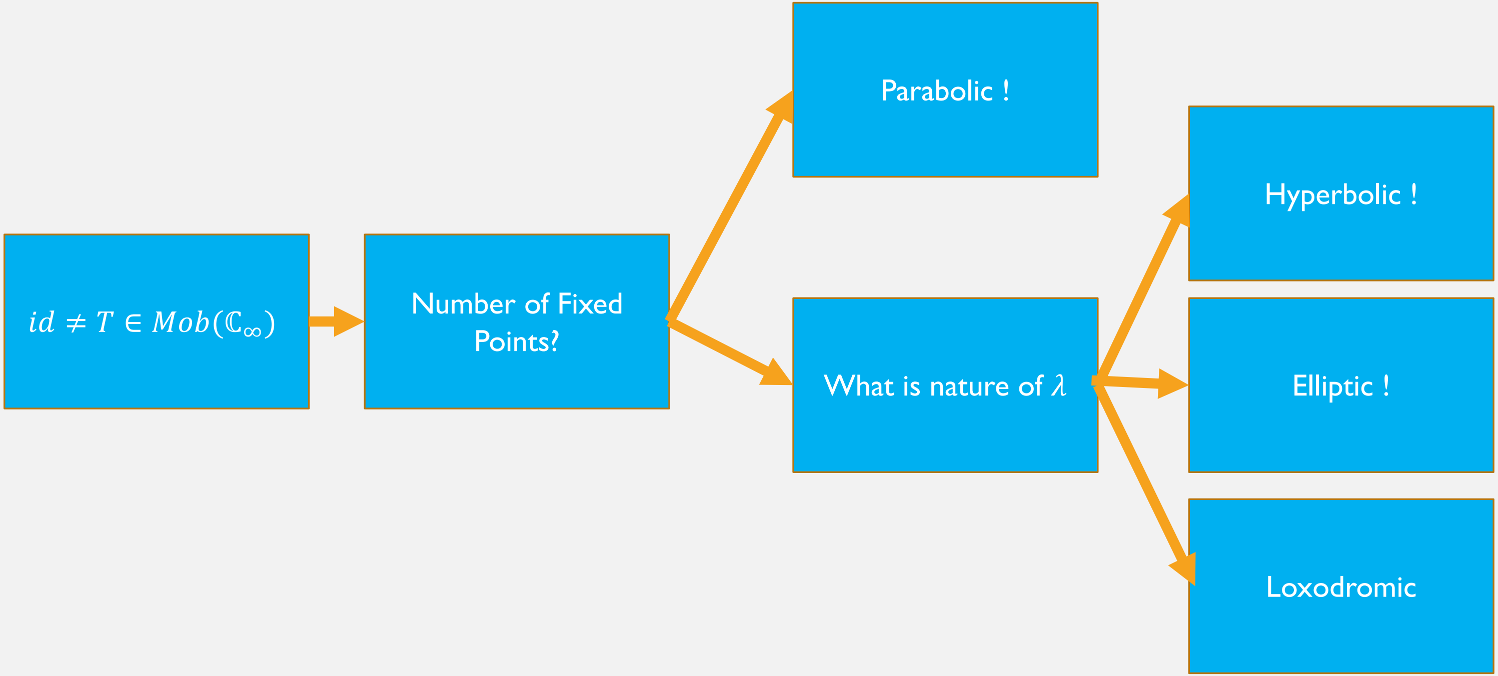


LOXODROMIC

- The remains cases are classified as *loxodromic*
- Write $\lambda = |\lambda| \times e^{\arg(\lambda)i}$, we can observe that *loxodromic* is just composition of *elliptic* transformation $e^{\arg(\lambda)i}z$ and *hyperbolic* transformation $|\lambda|z$



CLASSIFICATION SCHEME



CLASSIFICATION BY TRACE

- If $Tr(T) = \pm 2$ Then we immediately know that T is $parabolic$
- Similar thought applied to other cases !
- $T(z) = \lambda z = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda^{-1}} \end{pmatrix}$
- The value λ is called $multiplier$ of T

TRACE OF HYPERBOLIC MAP

- $Let\ l = \log(\lambda)$
- Then $Tr(T) = e^{\frac{l}{2}} + e^{-\frac{l}{2}} = 2\cosh\left(\frac{l}{2}\right)$
- $Hyperbolic \Leftrightarrow \lambda \in \mathbb{R}, \lambda \neq 1 \Leftrightarrow l = \mathbb{R} + 2n\pi i \Leftrightarrow |Tr(T)| > 2$

TRACE OF ELLIPTIC MAP

- $Let\ l = \log(\lambda)$
- Then $Tr(T) = e^{\frac{l}{2}} + e^{-\frac{l}{2}} = 2\cosh\left(\frac{l}{2}\right)$
- $Elliptic \Leftrightarrow |\lambda| = 1, \lambda \neq 1 \Leftrightarrow l = i\theta, \theta = \arg(\lambda) \Leftrightarrow Tr(T) = 2\cos(\theta) \Leftrightarrow Tr(T) \in (-2,2)$

SUMMARY

- Given any $id \neq T \in Mob(\mathbb{C}_\infty)$, we can classify it according to its trace.

Trace	Type
$Tr(T) = \pm 2$	*Parabolic*
$Tr(T) \in \mathbb{R}, \lvert Tr(T)\rvert > 2$	*Hyperbolic*
$Tr(T) \in \mathbb{R}, \lvert Tr(T)\rvert < 2$	*Elliptic*
$Tr(T) \notin \mathbb{R}$	*Loxodromic*

FROM $\mathbb{C}_\infty$ TO $D\ and\ H$

- $Mob(H) = \{T \in Mob(\mathbb{C}_\infty) : T(H) \subset H\}$
- $Mob(D) = \{T \in Mob(\mathbb{C}_\infty) : T(D) \subset D\}$
- Exercise: Verify that $Mob(H)$ and $Mob(D)$ are subgroups of $Mob(\mathbb{C})$
- The classification of $Mob(D), Mob(H)$ are easy if we know their matrix representation !

$Mob(D)$

Theorem

Every element of Möb($\mathbb{D}$) either has the form

$$p(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

or has the form,

$$p(z) = \frac{\alpha \overline{z} + \beta}{\overline{\beta} \overline{z} + \overline{\alpha}},$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$.

This result is proved by Team 2. The major application of this theorem is that $\forall T \in Mob(D), Tr(T) = 2Re(\alpha) \in \mathbb{R}$

MAP FROM D TO H

- Theorem 4:
- The map $C: z \to \frac{z-i}{z+i}$ on $\mathbb{C}_\infty$ satisfy $C(H) = D$
- Proof:
- $C(\infty) = 1, C(1) = -i, C(0) = -1$
- Hence, C map the circle of infinity ∂H to ∂D
- ∂D and ∂H separate C_∞ into two connected component
- Hence, $C(H) = D\ or\ C(H) = \mathbb{C}_\infty \backslash (D \cup \partial D)$
- $C(i) = 0$ implies $C(H) = D$

CAYLEY TRANSFORMATION

- Definition :
- $C: H \to D$ defined by $\mathrm{C}(\mathrm{z}) = \frac{z-i}{z+i}$ is called the Cayley transformation from H to D
- The map is well-defined by Theorem 4

MATRIX REPRESENTATION OF $Mob(H)$

- Theorem 5:
- $\forall F \in Mob(H), F$ has the representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in R$
- Proof:
- Define $h: H \to H \ by \ h(z) = \frac{z - Re(F(i))}{Im(F(i))}$
- Since $Im\big(h(z)\big) = \frac{Im(z)}{Im\big(F(i)\big)}$ and $Im(z) > 0, Im\big(F(i)\big) > 0$
- Hence, h is well-defined

THEOREM 5 (CONT.)

- Define $g: H \to H \; by \; g = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for some $\theta \in R$
- Suppose $z = x + yi \in H, then \; write \cos(\theta) = c, \sin(\theta) = s$
- $g(z) = \frac{cx+s+ysi}{c-xs-ysi} = \frac{1}{|c-xs-ysi|^2}(cx + s + ysi)(c - xs + ys)$
- So $Im\big(g(z)\big) = \frac{y}{|c-xs-ysi|^2}(c^2+s^2) > 0$
- Hence, g is well-defined.
- Define $T = g * h * F \in Mob(H)$

THEOREM 5 (CONT.)

- Direct computation yields, $T(i) = i, T'(i) = 1$
- Using Theorem 4, $A = CTC^{-1} \in Mob(D)$ and $A(0) = 0, A'(0) = 1$
- Since $A = \frac{az+b}{\bar{b}z+\bar{a}}$ implies $A' = \left(\frac{1}{\bar{b}z+\bar{a}}\right)^2 (|a|^2 - |b|^2)$
- As $|a|^2 - |b|^2 = 1, A'(0) = \frac{1}{\bar{a}^2} = 1.$ Combine $A(0) = 0, A'(0) = 1$
- We have $a = \pm 1, b = 0$
- So $A = id$
- By the invariant of fixed point, $T = id$ implies $F = h^{-1}g^{-1}$
- So matrix representation of F will be product of matrix of h^{-1}, g^{-1}

SUMMARY

$Mob(H)$	$Mob(D)$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1, a, b, c, d \in \mathbb{R}$	$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \lvert a\rvert^2 - \lvert b\rvert^2 = 1$
Hence, Trace of their representation is real !	

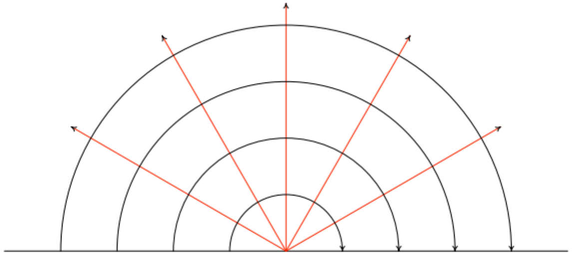
Trace	Type
$Tr(T) = \pm 2$	*Parabolic*
$Tr(T) \in R, \lvert Tr(T)\rvert > 2$	*Hyperbolic*
$Tr(T) \in R, \lvert Tr(T)\rvert < 2$	*Elliptic*

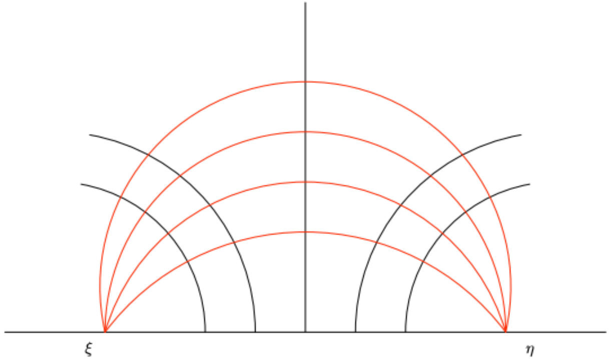
MORE ON HYPERBOLIC MAP

Hyperbolic map are conjugate to $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in \mathbb{R}$

So $xy = c$ $(Hyperbola)$ is an invariant curves under *Hyperbolic* map

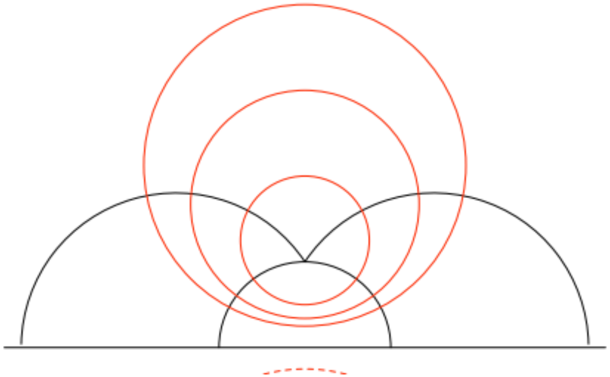
Also, its fixed points should location on ∂H or ∂D



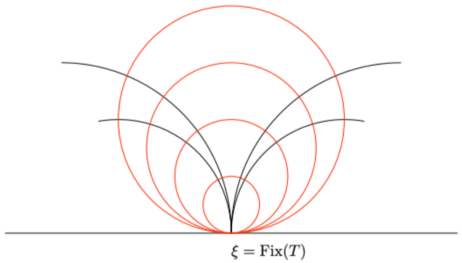


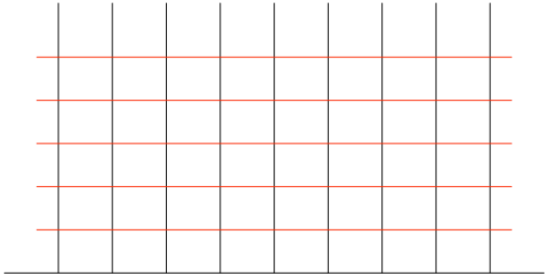
MORE ON ELLIPTIC MAP

- The conjugation with fixed point at 0 and ∞ is a rotation.
- $Circle$ will be an invariant curve.
- Fixed point will be located at interior of H or D



MORE ON PARABOLIC MAP





Exercise: Why this type of maps is called *Parabolic*?

The fixed point located at ∂H or ∂D

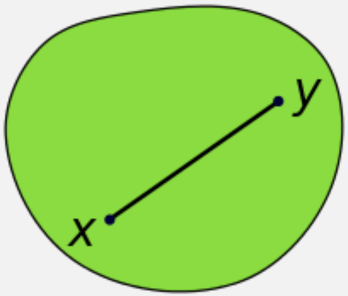
SUMMARY

- Matrix representation of group of Mobius Transformation
- Properties related to Conjugation, Fixed Point, Trace
- Classification of $Mob(\mathbb{C}_\infty)$ according to number of fixed point or trace
- Classification of $Mob(H), Mob(D)$

CONVEX SET IN H

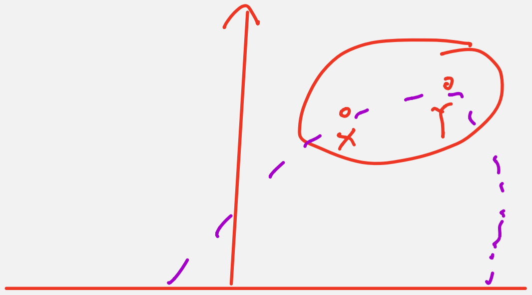
$Convex\ set\ in\ \mathbb{R}^n$

- A set C is $convex\ in\ R^n$ if $\forall x, y \in C, l_{xy} \subset C$
- Parametrization: $\forall x, y \in C, t \in [0,1], tx + (1-t)y \in C$



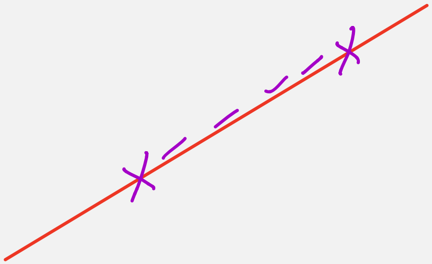
$Convex\ set\ in\ H$

- A set C is $convex$ in H if $\forall x, y \in C$, the hyperbolic line l_{xy} are contained in C



Line in $\mathbb{R}^n$	*Line in H*

- Theorem 1:
- Every Euclidean line is convex



- Theorem 1:
- Every Hyperbolic line is convex

Half Space in $\mathbb{R}^n$

- A line or an affine subspace of $\mathbb{R}^n$ are called *hyperplane*
- Hyperplane can be parametrized as $\mathrm{P} = \{< a, x > = c : x \in R^n\}$
- The hyperplane divide $\mathbb{R}^n$ into two separate connected component V_1, V_2
- V_1, V_2 is called open half-space
- $V_i \cup P$ is called closed half-space

Half Space in H

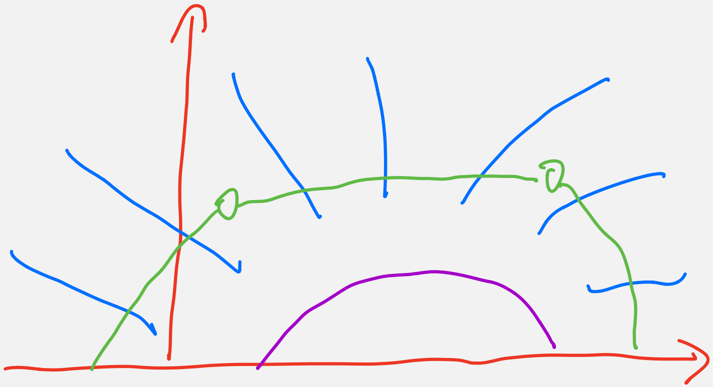
- Every hyperbolic line divide H into two connected component V_1, V_2
- V_1, V_2 is called open half-plane
- $V_i \cup P$ is called closed half-plane

$Convex\ Half\ Space\ in\ \mathbb{R}^n$

- Theorem 2:
- Every half-space is convex

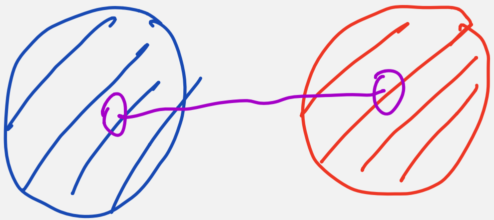
$Convex\ Half\ Space\ in\ H$

- Theorem 2:
- Every half-plane is convex



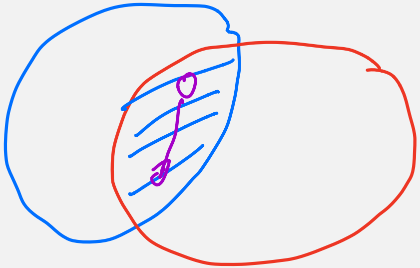
Operation on convex set in $\mathbb{R}^n$

- In general, intersection of convex set is convex
- While, union of convex set is not convex



Operation on convex set in H

- In general, intersection of convex set is convex
- While, union of convex set is not convex



$Projection\ in\ \mathbb{R}^n$

- Theorem 4:
- Let C be closed, convex set in $\mathbb{R}^n$, $z \in \mathbb{R}^n$.
- Then $\exists!\, x \in C, d(x,z) = d(z,C)$

$Projection\ in\ H$

- Theorem 4:
- Let C be closed, convex set in H, $z \in H$.
- Then $\exists!\, x \in C, d_H(x,z) = d_H(z,C)$

Idea:

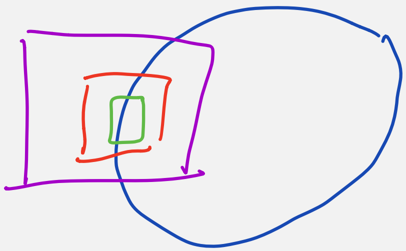
$d_H(z,C) = \inf\{d_H(z,x) : x \in C\}$

$\Rightarrow \exists\, (x_n) \subset C$ s.t. $d_H(z,x_n) \to d_H(z,C)$

+ Bolzano compactness.

CHARACTERIZATION OF CONVEX SET

- Theorem 5:
- Let $C \subset H$
- C is convex $\Leftrightarrow$ C is intersection of half-planes



REFERENCE

- Hyperbolic geometry, by James W. Anderson, Springer, 1999.
- Lecture notes by C. Series